Mathematics 232-B — Spring 2000

Exam 2

March 10, 2000

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Name

Technology used: ______________________________

Textbook/Notes used: ______________________________

Directions: Be sure to include in-line citations, including page numbers if appropriate, every time you use a text or notes or technology. Include a careful sketch of any graph obtained by technology in solving a problem. **Only write on one side of each page.**

The Problems

1. Do **one** of the following

   (a) We are told that a certain 5×5 matrix A can be written as

   $$A = BC$$

   where B is 5×4 and C is 4×5. Explain how you know that A is not invertible.

   (b) The definition of the orthogonal complement of a subspace V of $\mathbf{R}^n$ is the set, $V^\perp$, of all vectors in $\mathbf{R}^n$ that are perpendicular to every vector in V. Suppose that $\overrightarrow{v}_1, \ldots, \overrightarrow{v}_m$ is a basis for V. Show that $\overrightarrow{x} \in R^n$ is in $V^\perp$ if and only if $\overrightarrow{x}$ is orthogonal to each of the m basis vectors of V. That is, show the vector $\overrightarrow{x}$ satisfies

   $$\overrightarrow{x} \cdot \overrightarrow{v} = \overrightarrow{0} \text{ for every } \overrightarrow{v} \in R^n \quad \text{if and only if} \quad \overrightarrow{v}_1 \cdot \overrightarrow{x} = \overrightarrow{v}_2 \cdot \overrightarrow{x} = \cdots = \overrightarrow{v}_m \cdot \overrightarrow{x} = 0.$$

   (c) For two invertible $(n \times n)$ matrices A and B, determine which of the following formulas are **necessarily** true.

      i. $(A+B)^2 = A^2 + 2AB + B^2$.
      ii. $(A-B)(A+B) = A^2 - B^2$.
      iii. $ABB^{-1}A^{-1} = I_n$.
      iv. $ABA^{-1} = B$
      v. $\left(ABA^{-1}\right)^3 = AB^3A^{-1}$.

2. Do **one** of the following.

   (a) Is the set $W = \left\{ \vec{y} = \begin{bmatrix} 2x_1 - 3x_2 + x_3 \\ -x_1 + 3x_2 - 4x_3 \\ 5x_1 - 2x_2 \end{bmatrix} : x_1, x_2, x_3 \in R \right\}$ a subspace of R^3? Explain.

   (b) Let V be a subspace of R^n and let A be an $(m \times n)$ matrix. Is the set $W = \left\{ \overrightarrow{x} \in V : A\overrightarrow{x} = \overrightarrow{\theta} \right\}$ a subspace of R^n? Explain. [Note: W is **not** the null-space of A.]

3. Given the matrix $A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 1 & 2 & 6 & 3 & 4 \\ 1 & 3 & 9 & 5 & 2 \\ 1 & 4 & 12 & 7 & 0 \end{bmatrix}$, show that $\text{rank}(A) + \text{nullity}(A) = 5$ by computing both $\text{rank}(A)$ and $\text{nullity}(A)$.

4. Let A be an $(m \times m)$ non-singular matrix, and let B be an $(m \times n)$ matrix.

   (a) Prove that $N(AB) = N(B)$ (where $N(C)$ denotes the null-space of C)

   (b) Use part a. to prove that $\text{rank}(AB) = \text{rank}(B)$.

5. Use the Gram-Schmidt process to generate an orthogonal set from the given linearly independent vectors.

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$